CLASSICAL AND QUANTUM SYSTEMS

Foundations and Symmetries

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PATH INTEGRAL REALIZATION OF DYNAMICAL SYMMETRIES

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1 Introduction

Sixty years ago in his book [1] Eugen Wigner demonstrated the power of group theoretical methods in quantum mechanics. Without solving the Schrödinger equation many important quantum mechanical results can be obtained purely from symmetry considerations. Even group theory may provide exact solutions of the Schrödinger equation for certain systems. Most of the exactly soluble problems have been classified by the factorization method of Schrödinger, Infeld and Hull [2]. The factorization method is indeed related with Lie theory [3]. There are only two elementary systems to which all others can be reduced by changing variables and transforming the wavefunction in the Schrödinger equation. The two systems are the radial harmonic oscillator having solutions of confluent hypergeometric type and the Pöschl-Teller oscillator possessing solutions of hypergeometric type. The underlying group structures of the exactly soluble problems are associated with symmetries of dynamical origin rather than geometrical ones.

The aim of the present report is to demonstrate that group theory is also very useful in the path integral approach to quantum mechanics [4]. The application of group theory to path integrals with geometrical symmetries has already been reported in the last Wigner Symposium [5]. Here we wish to discuss the path integral realization of dynamical symmetries. As an example we shall consider the radial path integral of a harmonic oscillator in $\mathbb{R}^d$. First, we shall explicitly demonstrate that this system has the $SU(1,1)$ dynamical group. Then we shall present its path integral representation. For the realization of dynamical symmetries $SU(2)$ and $SU(1,1)$ of the Pöschl-Teller systems we refer to refs. [6-8]. In particular, for the local time rescaling technique, which is equivalent to the change of variable and transformation of wavefunction in the Schrödinger approach, see refs. [8,9].

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2 The dynamical group $SU(1,1)$ of the harmonic oscillator in $\mathbb{R}^d$

The Lie algebra of $SU(1,1)$ may explicitly be given by the commutators

$$[J_1,J_2]=-\mathrm{i}J_3,\quad [J_2,J_3]=\mathrm{i}J_1,\quad [J_3,J_1]=\mathrm{i}J_2. \tag{1}$$

The quadratic Casimir operator is $\mathbf{J}^2=-J_1^2-J_2^2+J_3^2$. In a given unitary irreducible representation (UIR) it is proportional to the unit operator, i.e. $\mathbf{J}^2=J(J+1)\mathbf{1}$, where the "angular momentum" quantum number J may be used for labelling all UIR's. There are two continuous and two discrete series of UIR [10]. Spectra of the compact operator J_3 for the continuous series are unbounded. Whereas, the spectra of J_3 for the discrete series denoted by D_J^- and D_J^+ are bounded from above and below, respectively. It is the series D_J^+ which is realized by the harmonic oscillator. On the standard discrete basis we have for D_J^+ [10]:

$$\mathbf{J}^2|J,m\rangle = J(J+1)|J,m\rangle \quad \text{with} \quad -1<J, \tag{2}$$
$$J_3|J,m\rangle = m|J,m\rangle \quad \text{with} \quad m=J+1,J+2,\dots\,. \tag{3}$$

It is also possible to choose a continuous basis where a noncompact operator is diagonalized [11]. The one we are interested in is $K:=J_1+J_3$ and has the positive real line as the spectrum for D_J^+:

$$K|J,\eta\rangle=\eta|J,\eta\rangle \quad \text{with} \quad \eta\in\mathbb{R}^+. \tag{4}$$

For the matrix element for a finite transformation in the continuous basis we find [11] in analogy to Wigner's d-function [1]

$$\langle J,\eta|\mathrm{e}^{-2\mathrm{i}\varphi J_3}|J,\eta'\rangle=\frac{1}{\mathrm{i}\sin\varphi}\exp\{\mathrm{i}(\eta+\eta')\cot\varphi\}\ \mathrm{I}_{2J+1}\left(\frac{2\sqrt{\eta\eta'}}{\mathrm{i}\sin\varphi}\right) \tag{5}$$

where $\mathrm{I}_\nu(z)$ is the modified Bessel function, $0<\varphi<\pi$ and $\eta,\eta'\in\mathbb{R}^+$.

In the following we consider a creation and an annihilation operator for each degree of freedom in $\mathbb{R}^d$, that is $[a_i,a_j^\dagger]=\delta_{ij}$ $(i,j=1,\dots,d)$. A realization of the algbra of $SU(1,1)$ is given by

$$J_1:=\frac{1}{4}\sum_{i=1}^d\left(a_i^{\dagger\,2}+a_i^2\right),\quad J_2:=-\frac{\mathrm{i}}{4}\sum_{i=1}^d\left(a_i^{\dagger\,2}-a_i^2\right),\quad J_3:=\frac{1}{2}\sum_{i=1}^d\left(a_i^\dagger a_i+\frac{1}{2}\right). \tag{6}$$

The Hamiltonian for the harmonic oscillator in $\mathbb{R}^d$ is $H=2\hbar\omega J_3$ which is bounded below (i.e. D_J^+). We observe that the total angular momentum associated with $SO(d)$ in $\mathbb{R}^d$ is related to the Casimir operator (2) by $\mathbf{L}^2=4\mathbf{J}^2-d(d-4)/4$ and has eigenvalues $l(l+d-2)$, $l\in\mathbb{N}_0$. Therefore, a fixed angular momentum subspace corresponds to the representation D_J^+ with $J:=l/2+d/4-1$. The spectrum (3) of J_3 leads to the energy eigenvalues $E_n:=\hbar\omega(2n+l+d/2)$, $n\in\mathbb{N}_0$. Finally, we note that the time evolution operator $\exp\{-(\mathrm{i}/\hbar)H\tau\}=\exp\{-2\mathrm{i}\omega\tau J_3\}$ is a group element and that $SU(1,1)$ is indeed the *dynamical group* of the system.

3 Path integral realization of the dynamical group $SU(1,1)$

According to Feynman [4] the propagator, i.e. the matrix element of the time evolution operator, may be given by a path integral,

$$\langle \mathbf{x}''|\mathrm{e}^{-(i/\hbar)H\tau}|\mathbf{x}'\rangle = \int_{\mathbf{x}'=\mathbf{x}(0)}^{\mathbf{x}''=\mathbf{x}(\tau)} \mathcal{D}[x(t)] \exp\left\{\frac{\mathrm{i}}{\hbar}\int_0^\tau \left(\frac{M}{2}\dot{\mathbf{x}}^2 - \frac{M}{2}\omega^2\mathbf{x}^2\right)\mathrm{d}t\right\},$$

which reads on the sliced time basis $[\varepsilon := \tau/N,\ \mathbf{x}_j := \mathbf{x}(j\varepsilon)]$

$$\langle \mathbf{x}''|\mathrm{e}^{-(i/\hbar)H\tau}|\mathbf{x}'\rangle = \lim_{N\to\infty}\left(\frac{M}{2\pi\mathrm{i}\hbar\varepsilon}\right)^{Nd/2} \int \prod_{j=1}^{N} \exp\left\{\frac{\mathrm{i}}{\hbar}\left[\frac{M}{2\varepsilon}(\mathbf{x}_j-\mathbf{x}_{j-1})^2 - \frac{M}{4}\omega^2\varepsilon(\mathbf{x}_j^2+\mathbf{x}_{j-1}^2)\right]\right\} \prod_{j=1}^{N-1}\mathrm{d}\mathbf{x}_j. \tag{7}$$

Due to its spherical symmetry we immediately can perform the angular path integration [7]:

$$\langle \mathbf{x}''|\mathrm{e}^{-(i/\hbar)H\tau}|\mathbf{x}'\rangle = \sum_{l=0}^{\infty} K_l(r'',r';\tau)\sum_{\mathcal{M}}\frac{\Gamma(d/2)}{2\pi^{d/2}}Y_{l\mathcal{M}}(\mathbf{x}''/r'')Y^*_{l\mathcal{M}}(\mathbf{x}'/r'), \tag{8}$$

where $Y_{l\mathcal{M}}(\mathbf{e})$ are the hyperspherical harmonics in $\mathbb{R}^d$ and $\mathcal{M}$ stands for a $(d-2)$-tuple counting the degeneracy of the angular momentum l. The radial propagator is given by the remaining path integral $[r_j := |\mathbf{x}_j|]$

$$K_l(r'',r';\tau) = \left(\frac{1}{r''r'}\right)^{(d-2)/2} \lim_{N\to\infty}\int\prod_{j=1}^{N}R(r_j,r_{j-1})\prod_{j=1}^{N-1}r_j\mathrm{d}r_j \tag{9}$$

where

$$R(r_j,r_{j-1}) := \frac{M}{\mathrm{i}\hbar\varepsilon}\exp\left\{\frac{\mathrm{i}}{\hbar}\left(r_j^2+r_{j-1}^2\right)\left(\frac{M}{2\varepsilon}-\frac{M}{4}\omega^2\varepsilon\right)\right\}\mathrm{I}_{l+(d-2)/2}\left(\frac{Mr_jr_{j-1}}{\mathrm{i}\hbar\varepsilon}\right). \tag{10}$$

It is the dynamical group $SU(1,1)$ which now enables us to complete the path integration. First, we set $\eta_j := M\omega r_j^2/2\hbar$ which gives for the exponential in (10)

$$\exp\{\cdots\} = \exp\{\mathrm{i}(\eta_j+\eta_{j-1})(1/\omega\varepsilon - \omega\varepsilon/2)\}. \tag{11}$$

Note also that $r_j\mathrm{d}r_j = (\hbar/M\omega)\mathrm{d}\eta_j$. Secondly, we define $\sin\varphi := \omega\varepsilon$ which leads to

$$1/\omega\varepsilon - \omega\varepsilon/2 = \cot\varphi + \mathcal{O}(\varepsilon^3). \tag{12}$$

Inserting this in (11) [note that terms of $\mathcal{O}(\varepsilon^3)$ may be ignored in path integration] we find

$$\frac{\hbar}{M\omega}R(r_j,r_{j-1}) = \frac{1}{\mathrm{i}\sin\varphi}\exp\{\mathrm{i}(\eta_j+\eta_{j-1})\cot\varphi\}\,\mathrm{I}_{l+(d-2)/2}\left(\frac{2\sqrt{\eta_j\eta_{j-1}}}{\mathrm{i}\sin\varphi}\right). \tag{13}$$

Now we observe that this is identical with the matrix element (5) if we set for the representation label $J := l/2 + d/4 - 1$. Therefore, the radial path integral reads

$$K_l(r'',r';\tau) = \frac{M\omega}{\hbar}\left(\frac{1}{r''r'}\right)^{(d-2)/2}\lim_{N\to\infty}\int\prod_{j=1}^{N}\langle J,\eta_j|\mathrm{e}^{-2\mathrm{i}\varphi J_3}|J,\eta_{j-1}\rangle\prod_{j=1}^{N-1}\mathrm{d}\eta_j \tag{14}$$

and is easily performed using the completeness relation $\int_0^\infty \mathrm{d}\eta_j\,|J,\eta_j\rangle\langle J,\eta_j| = \mathbf{1}$ for the continuous basis (4). Finally, the radial propagator is given by

$$K_l(r'',r';\tau) = \frac{M\omega}{\hbar}\left(\frac{1}{r''r'}\right)^{(d-2)/2}\langle J,\eta''|\mathrm{e}^{-2\mathrm{i}\Phi J_3}|J,\eta'\rangle \tag{15}$$

where $\eta := M\omega r^2/2\hbar$ and

$$\Phi := \lim_{N\to\infty}[N\varphi] = \lim_{N\to\infty}[N\arcsin(\omega\tau/N)] = \omega\tau. \tag{16}$$

The present path integral treatment is an alternative to the earlier approach [12]. Besides its simplicity the present group theoretical approach has the advantage to realize the dynamical group $SU(1,1)$ explicitly [see eq. (15)]. It can also be applied to the radial path integral for the $1/r$-problem in $\mathbb{R}^d$ and the generalized Morse potential in $\mathbb{R}$, both of which have the dynamical group $SU(1,1)$ [8].

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